N89-15186

Appendix III

III.1 Strain Energy Release Rate

In this analysis, a delamination between belt and core sections is assumed to grow parallel to the belt direction in the tapered and uniform sections. These delaminations in each section are denoted by a and b respectively. The core section in the taper portion is modelled by two equivalent sublaminates. The stiffness properties are smeared to obtain the effective cracked and uncracked stiffnesses which are designated by A_u and A_c as shown in Figure III.1. These stiffnesses change from one ply drop group to another with crack growth a by experiencing a sudden change at discrete locations. Therefore A_u and A_c can be represented in three consecutive regions as follows,

- Region 1: $0 < a < l$

$$A_u = \frac{d+3l-a}{\frac{d}{A_{BD}} + \frac{l}{A_1} + \frac{l}{A_2} + \frac{l-a}{A_3}} \tag{III.1}$$

$$A_c = A_3 \tag{III.2}$$

- Region 2: $l < a < 2l$

$$A_u = \frac{d+3l-a}{\frac{d}{A_{BD}} + \frac{l}{A_1} + \frac{2l-a}{A_2}} \tag{III.3}$$

$$A_c = \frac{a+b}{\frac{a-l}{A_2} + \frac{l+b}{A_3}} \tag{III.4}$$

- Region 3: $2l < a < 3l$

$$A_u = \frac{d + 3l - a}{\frac{d}{A_{BD}} + \frac{l}{A_1}} \tag{III.5}$$

$$A_c = \frac{a + b}{\frac{a-2l}{A_1} + \frac{l}{A_2} + \frac{l+b}{A_3}} \tag{III.6}$$

where

$h =$ ply thickness

$d =$ length of uniform thick portion

$l =$ distance between two consecutive ply drop locations

$A_1 = 6hQ^{45} + 2hQ^0$

$A_2 = 4hQ^{45} + 2hQ^0$

$A_3 = 2hQ^{45} + 2hQ^0$

$A_{BD} = 7hQ^0 + 2hQ^{45}$

$Q^0 = Q_{11}$ of a 0 degree ply

$Q^{45} = Q_{11}$ of a ± 45 degree ply

Geometry of the sublaminate model is shown in Figure (III.1)
Also axial stiffnesses A_B, A_s and A_F are given by

$$A_B = \frac{d + 3l - a}{\frac{d}{A_{BD}} + \frac{3l-a}{A_{BT}}} \tag{III.7}$$

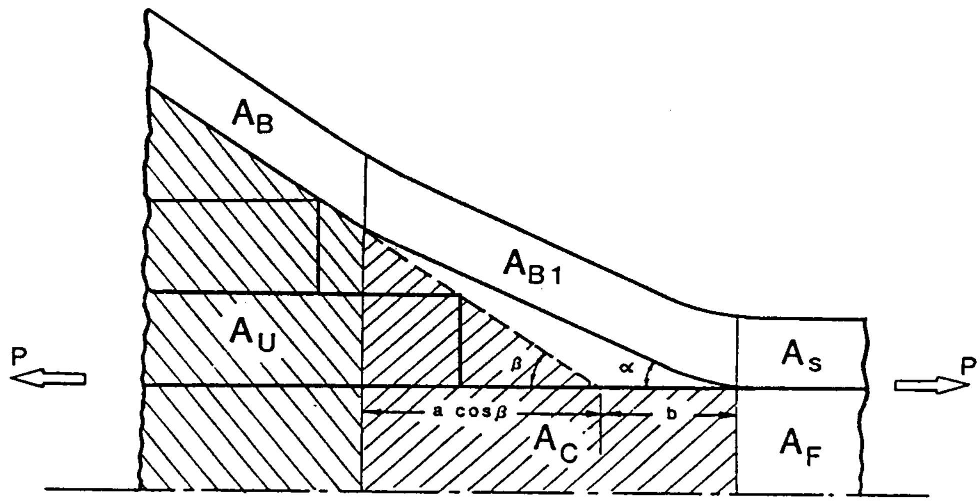

Figure III.1: Geometry of the Sublaminate Model

$$A_F = A_3 \quad \text{(III.8)}$$

$$A_s = A_{BD} \quad \text{(III.9)}$$

where

A_{BT} =Taper belt stiffness

For a membrane behavior, equilibrium equations are reduced to

$$N_{,x} = 0 \quad \text{(III.10)}$$

and the displacement field is assumed to be

$$u(x,z) = U(x) \quad \text{(III.11)}$$

and

$$w = 0 \quad \text{(III.12)}$$

The constitutive relations are represented by

$$N = A_{11} U_{,x} \tag{III.13}$$

The stress and displacement fields, are determined based on the stiffnesses derived in Equations(III.1-III.9). In this model, load is shared by the core and the belt portions according to their respective stiffness ratios

$$P_1 = \frac{P A_B}{A_B + A_u} \tag{III.14}$$

$$P_2 = \frac{P A_u}{A_B + A_u} \tag{III.15}$$

where P is half of the total axial load applied at the ends.

Using the Equations (III.10), (III.13), and the expressions for P_1 and P_2 from Equations (III.14), (III.15) the axial displacement at $x = c$ can be written as

$$\begin{aligned} U_5 &= \frac{P A_B c}{A_s(A_B + A_u)} + \frac{P(d + 3l + b)}{(A_B + Au)} \left(\frac{A_B}{A_{B1}} - \frac{A_B}{A_s} \right) \\ &+ \frac{P(d + 3l - a)}{(A_B + Au)} \left(1 - \frac{A_B}{A_{B1}} \right) \end{aligned} \tag{III.16}$$

$$\begin{aligned} U_6 &= \frac{P A_u c}{A_3(A_B + A_u)} + \frac{P(d + 3l + b)}{(A_B + Au)} \left(\frac{A_u}{A_c} - \frac{A_u}{A_F} \right) \\ &+ \frac{P(d + 3l - a)}{(A_B + Au)} \left(1 - \frac{A_u}{A_c} \right) \end{aligned} \tag{III.17}$$

where A_{B1} is the belt stiffness in the pop-off region as shown in Figure III.1.

A three-dimensional transformation is required in order to estimate the effective axial stiffness of the belt region A_B and A_{B1}. This is due to the belt layup and the orientation of the different belt portions to the loading axis as shown in Figure III.1. The three-dimensional transformation is presented in section III.3.

The tapered laminate is assumed to be fixed at $x = 0$. Therefore the external work done is given by

$$W = P_1 U_5 + P_2 U_6 \tag{III.18}$$

Substitute from Equations (III.14) through (III.17) into Equation (III.18) to get

$$\frac{W}{P^2} = \frac{1}{(A_B + A_u)^2} \left[(c - d - 3l - b)\left(\frac{A_B^2}{A_s} + \frac{A_u^2}{A_F}\right) + (d + 3l - a)(A_B + A_u) + (a + b)\left(\frac{A_B^2}{A_{B1}} + \frac{A_u^2}{A_c}\right)\right] \tag{III.19}$$

The strain energy release rate G due to the external work done is determined by

$$G = \frac{1}{2P^2}\frac{dW}{dA} \tag{III.20}$$

where A is the delamination surface area. G is calculated for delamination lengths ranging from 0 to $60h$. In the analysis, S2/SP250 Glass-Epoxy is used. Its properties are given in Table III.1.

Table III.1: Material Properties of S2/SP250 Glass-Epoxy

E_{11} (MSI)	E_{22} (MSI)	G_{12} (MSI)	G_{13} (MSI)	G_{23} (MSI)	ν_{12}
7.3	2.1	0.87	0.5	0.5	0.275

III.2 Interlaminar Stresses

In this part, an analysis for the interlaminar stresses in the belt-core interface in the tapered section will is developed.

The simple analytical model assumes a beam model for the belt in the tapered section which is shown in Figure III.2 . Material and geometric discontinuities are modelled as extensional k_i and concentrated shear springs g_i (i=1-4) as shown in Figure III.3. The resin pockets are assumed to be subjected primarily to shear stress and they are represented by a distributed shear spring with a constant stiffness G. The effect of the core is incorporated as elastic supports on the beam-belt model.

A minimum complementary potential energy formulation is used to estimate the interlaminar stresses. The total complementary potential energy consists of bending, shear and extensional energy contributions,

$$\Pi^c = \Pi_b + \Pi_s + \Pi_e + \Pi_k \tag{III.21}$$

where Π_b, Π_s, Π_e, Π_k represent bending, shear and extensional energy components and energy stored in elastic springs, respectively. These are given as,

$$\Pi_b = \frac{1}{2}\int_0^{3l} \frac{M^2(s)}{D_{11}} ds \tag{III.22}$$

$$\Pi_s = \frac{1}{2}\int_0^{3l} \frac{\alpha V^2(s)}{G_1} ds \tag{III.23}$$

$$\Pi_e = \frac{1}{2}\int_0^{3l} \frac{N^2(s)}{A_{11}} ds \tag{III.24}$$

$$\Pi_k = \frac{1}{2}\int_0^{3l} \frac{\tau^2(s)}{G_2} ds + \frac{R_1^2}{2k_1} + \frac{R_2^2}{2k_2} + \frac{R_3^2}{2k_3} + \frac{R_4^2}{2k_4} + \frac{T_1^2}{2g_1} + \frac{T_2^2}{2g_2} + \frac{T_3^2}{2g_3} + \frac{T_4^2}{2g_4} \tag{III.25}$$

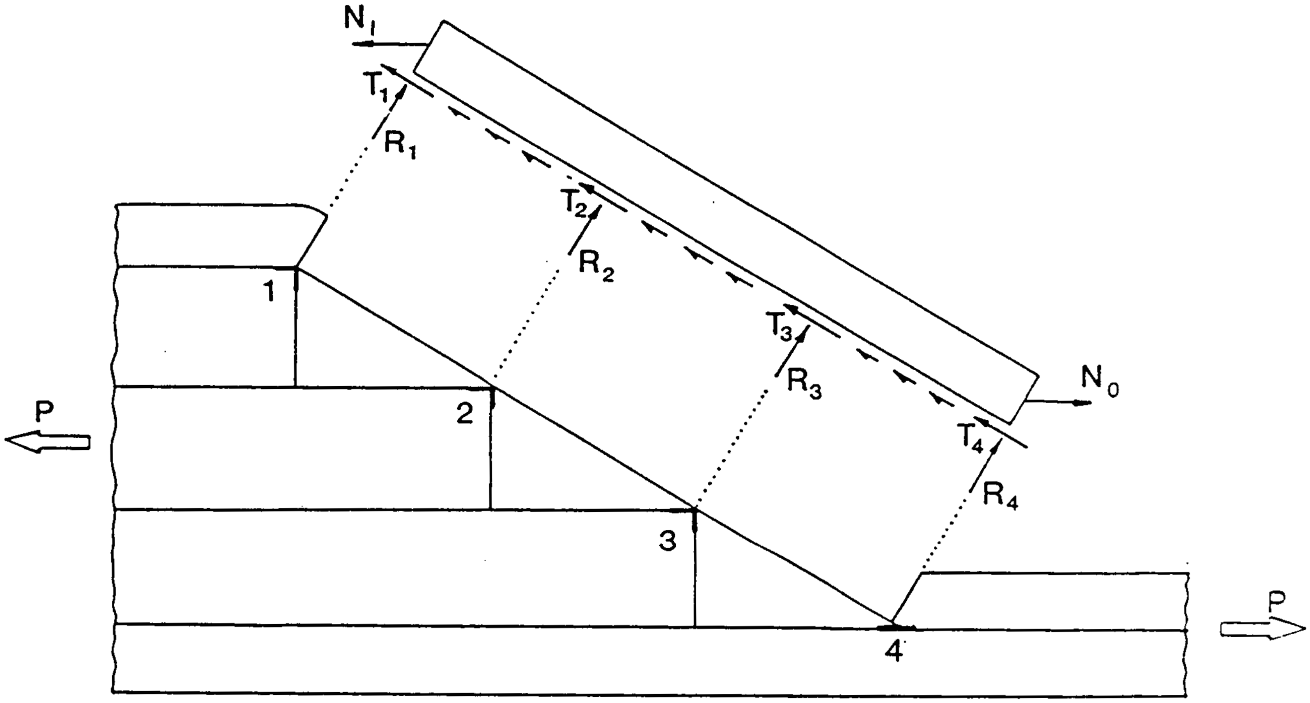

Figure III.2: Geometry of the Model

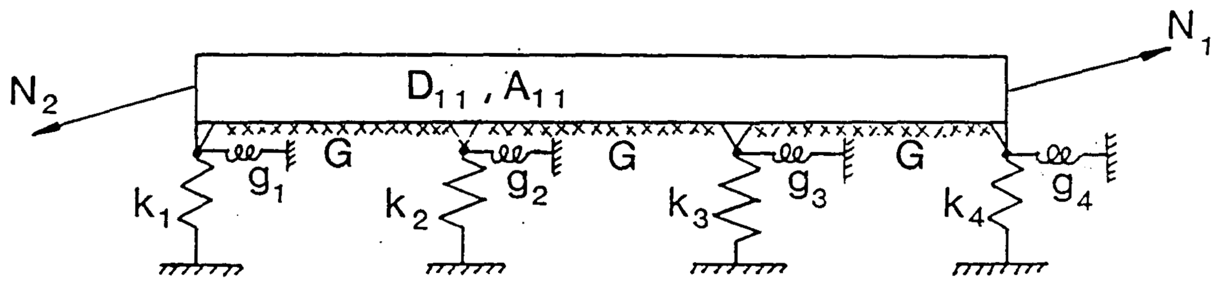

Figure III.3: Modelling of the Beam-Belt where R_i, T_i (i=1,2,3,4) are unknowns. The constant shear stress,c, due to resin filler is an additional unknown. The total number of unknowns in this formulation is nine. These unknowns are constrained by following equilibrium equations.

$$R_1 = -R_3 - 2R_4 + 2N_{12} + N_{22} - \frac{t}{2l}(N_{11} - N_{21}) \tag{III.26}$$

$$R_2 = 2R_3 + 3R_4 - 3N_{12} + \frac{t}{2l}(N_{11} - N_{21}) \tag{III.27}$$

$$T_1 = -T_2 - T_3 - T_4 - 3cl - N_{12} + N_{11} \tag{III.28}$$

where N_{11}, N_{12}, N_{21} and N_{22} denote the components of the extensional load at two ends of the belt section.

The bending moment, shear force and axial force in each of the three ply drop regions are written as

- Region 1: $0 < s < l$

$$M(s) = -N_{12}s + \frac{ct}{2}s + R_4 s + T_4\frac{t}{2} \tag{III.29}$$

$$V(s) = N_{12} - R_4 \tag{III.30}$$

$$N(s) = N_{11} - cs - T_4 \tag{III.31}$$

- Region 2: $l < s < 2l$

$$M(s) = -N_{12}s + \frac{ct}{2}s + (R_3 + R_4)s - R_3 l + (T_3 + T_4)\frac{t}{2} \tag{III.32}$$

$$V(s) = N_{12} - R_3 - R_4 \tag{III.33}$$

$$N(s) = N_{11} - cs - T_3 - T_4 \tag{III.34}$$

- Region 3: $2l < s < 3l$

$$M(s) = -N_{12}s + \frac{ct}{2}s + (-R_2 + R_3 + R_4)s + (2R_2 - R_3)l + (T_2 + T_3 + T_4)\frac{t}{2} \tag{III.35}$$

$$V(s) = N_{12} - R_2 - R_3 - R_4 \tag{III.36}$$

$$N(s) = N_{11} - cs - T_2 - T_3 - T_4 \tag{III.37}$$

Therefore the bending energy in Equation (III.22) can be written as

$$\begin{aligned} \Pi_b \quad = \quad & \frac{1}{2D_{11}} \int_0^l \left[-N_{12}s + \frac{ct}{2}s + R_4 s + T_4 \frac{t}{2} \right]^2 ds \\ & + \frac{1}{2D_{11}} \int_l^{2l} \left[-N_{12}s + \frac{ct}{2}s + (R_3 + R_4)s - R_3 l + (T_3 + T_4)\frac{t}{2} \right]^2 ds \\ & + \frac{1}{2D_{11}} \int_{2l}^{3l} \left\{ -N_{12}s + \frac{ct}{2}s + \left[-R_3 - 2R_4 + N_{12} - \frac{t}{2l}(N_{11} - N_{21}) \right] s \right. \end{aligned}$$

$$+\left[3R_3 + 6R_4 - 6N_{12} + \frac{t}{2l}(N_{11} - N_{21})\right] l$$

$$+(T_2 + T_3 + T_4)\frac{t}{2}\Big\}^2 ds \qquad \text{(III.38)}$$

Similarly for the shear energy

$$\begin{aligned}\Pi_s &= r\int_0^l (N_{12} - R_4)^2\, ds + r\int_l^{2l} (N_{12} - R_3 - R_4)^2\, ds \\ &\quad + r\int_{2l}^{3l} \left[-2N_{12} + R_3 + 2R_4 + \frac{t}{2l}(N_{11} - N_{21})\right]^2 ds \qquad \text{(III.39)}\end{aligned}$$

where

$$r = \tfrac{3}{5G_2A} \ .$$

The energy of extensional loads can be expressed by

$$\begin{aligned}\Pi_e &= \frac{1}{2A_{11}}\int_0^l (N_{11} - cs - T_4)^2\, ds + \frac{1}{2A_{11}}\int_l^{2l} (N_{11} - cs - T_3 - T_4)^2\, ds \\ &\quad + \frac{1}{2A_{11}}\int_{2l}^{3l} (N_{11} - cs - T_2 - T_3 - T_4)^2\, ds \qquad \text{(III.40)}\end{aligned}$$

The energy stored in the elastic springs is written as

$$\begin{aligned}\Pi_k &= \frac{3}{2}\frac{c^2}{G_2}l + \frac{1}{2k_1}\left[-R_3 - 2R_4 + 2N_{12} + N_{22} - \frac{t}{2l}(N_{11} - N_{21})\right]^2 \\ &\quad + \frac{1}{2k_2}\left[2R_3 + 3R_4 - \frac{1}{l}M_1 + \frac{1}{l}M_2 - 3N_{12} + \frac{t}{2l}(N_{11} - N_{21})\right]^2\end{aligned}$$

$$+\frac{R_3^2}{2k_3}+\frac{R_4^2}{2k_4}+\frac{T_1^2}{2g_1}+\frac{T_2^2}{2g_2}+\frac{1}{2g_1}\left[-T_2-T_3-T_4-3cl-N_{12}+N_{11}\right]^2$$

$$+\frac{T_2^2}{2g_2}+\frac{T_3^2}{2g_3}+\frac{T_4^2}{2g_4} \tag{III.41}$$

The complementary potential energy in Equations (III.38) through (III.41) is expressed in terms of 6 unknowns, namely R_3, R_4, T_i (i=2,3,4) and c. By minimizing these expressions the following system of linear equations is obtained

$$\left(\frac{27t^2l^3}{12D}+\frac{3l}{G_2}+\frac{9l^2}{g_1}\right)c+\frac{tl^3}{D}R_3+\frac{5tl^3}{2D}R_4+\left(\frac{5t^2l^2}{8D}+\frac{3l}{g_1}\right)T_2+\left(\frac{t^2l^2}{D}+\frac{3l}{g_1}\right)T_3$$

$$+\left(\frac{9t^2l^2}{8D}+\frac{3l}{g_1}\right)T_4=\frac{5tl^3}{2D}N_{12}+\frac{3l}{g_1}(N_{11}-N_{21})+\frac{t^2l^2}{3D}(N_{11}-N_{21}) \tag{III.42}$$

$$\frac{tl^3}{D}c+\left(\frac{2l^3}{3D}+\frac{1}{k_1}+\frac{4}{k_2}+\frac{1}{k_3}\right)R_3+\left(\frac{3l^3}{2D}+\frac{2}{k_1}+\frac{6}{k_2}\right)R_4+\frac{tl^2}{4D}T_2$$

$$+\frac{tl^2}{2D}T_3+\frac{tl^2}{2D}T_4=\left(\frac{3l^3}{2D}+\frac{2}{k_1}+\frac{6}{k_2}\right)N_{12}+\frac{2N_{22}}{k_1}$$

$$+\left(\frac{tl^3}{12D}-\frac{t}{2k_1l}-\frac{t}{k_2l}\right)(N_{11}-N_{21}) \tag{III.43}$$

$$\frac{5tl^3}{2D}c+\left(\frac{3l^3}{2D}+\frac{2}{k_1}+\frac{6}{k_2}\right)R_3+\left(\frac{4l^3}{D}+\frac{4}{k_1}+\frac{9}{k_2}+\frac{1}{k_4}\right)R_4+\frac{tl^2}{2D}T_2+\frac{5tl^2}{4D}T_3$$

$$+\frac{3tl^2}{2D}T_4=\left(\frac{4l^3}{D}+\frac{4}{k_1}+\frac{9}{k_2}\right)N_{12}+\frac{2N_{22}}{k_1}+\frac{tl^2}{6D}(N_{11}-N_{21}) \tag{III.44}$$

$$\left(\frac{5t^2l^2}{8D}+\frac{3l}{g_1}\right)c+\frac{tl^2}{4D}R_3+\frac{tl^2}{2D}R_4+\left(\frac{t^2l}{4D}+\frac{1}{g_1}+\frac{1}{g_2}\right)T_2+\left(\frac{t^2l}{4D}+\frac{1}{g_1}\right)T_3$$

$$+\left(\frac{t^2l}{4D}+\frac{1}{g_1}\right)T_4=\frac{tl^2}{2D}N_{12}+\frac{1}{g_1}(N_{11}-N_{21})+\frac{t^2l}{8D}(N_{11}-N_{21}) \qquad \text{(III.45)}$$

$$\left(\frac{t^2l^2}{D}+\frac{3l}{g_1}\right)c+\frac{tl^2}{2D}R_3+\frac{5tl^2}{4D}R_4+\left(\frac{t^2l}{4D}+\frac{1}{g_1}\right)T_2+\left(\frac{t^2l}{2D}+\frac{1}{g_1}+\frac{1}{g_3}\right)T_3$$

$$+\left(\frac{t^2l}{2D}+\frac{1}{g_1}\right)T_4=\frac{5tl^2}{4D}N_{12}+\frac{1}{g_1}(N_{11}-N_{21})+\frac{t^2l}{8D}(N_{11}-N_{21}) \qquad \text{(III.46)}$$

$$\left(\frac{9t^2l^2}{8D}+\frac{3l}{g_1}\right)c+\frac{tl^2}{2D}R_3+\frac{3tl^2}{2D}R_4+\left(\frac{t^2l}{4D}+\frac{1}{g_1}\right)T_2+\left(\frac{t^2l}{2D}+\frac{1}{g_1}\right)T_3$$

$$+\left(\frac{3t^2l}{4D}+\frac{1}{g_1}+\frac{1}{g_4}\right)T_4=\frac{3tl^2}{2D}N_{12}+\frac{1}{g_1}(N_{11}-N_{21})$$

$$+\frac{t^2l}{16D}(N_{11}-N_{21}) \qquad \text{(III.47)}$$

The concentrated normal and shear forces at the ply drop regions and the interlaminar shear in the resin filler are estimated by solving the simultaneous system of equations in (III.42) through (III.47) and using Equation (III.26) through (III.28).

III.3 3-D Transformation of Stiffnesses

It has been determined that a three dimensional transformation of stiffnesses is required in order to estimate the effective axial stiffness of the belt regions, A_B and A_{B1}. This is due to the belt layup and the orientation of the different belt portions to the loading axis as shown in Figure III.4.

The loading axis corresponds to axis 1 in the 123 coordinate system which is the transformed system. The principal material coordinates are denoted by $1'$,$2'$ and $3'$.

The stress-strain relationships in the principal material coordinates for an orthotropic laminate are given by

$$\{\overline{\sigma}\}_{6\times1} = [Q]_{6\times6}\{\overline{\epsilon}\}_{6\times1} \tag{III.48}$$

where

$$Q_{11} = (1-\nu_{23}\nu_{32})VE_{11} \tag{III.49}$$

$$Q_{22} = (1-\nu_{31}\nu_{13})VE_{22} \tag{III.50}$$

$$Q_{33} = (1-\nu_{12}\nu_{21})VE_{33} \tag{III.51}$$

$$Q_{12} = (\nu_{21}+\nu_{23}\nu_{31})VE_{11} = (\nu_{12}+\nu_{13}\nu_{32})VE_{22} \tag{III.52}$$

$$Q_{13} = (\nu_{31}+\nu_{21}\nu_{32})VE_{11} = (\nu_{13}+\nu_{23}\nu_{12})VE_{33} \tag{III.53}$$

$$Q_{23} = (\nu_{32}+\nu_{12}\nu_{31})VE_{22} = (\nu_{23}+\nu_{21}\nu_{13})VE_{33} \tag{III.54}$$

$$Q_{44} = G_{23} \tag{III.55}$$

$$Q_{55} = G_{31} \tag{III.56}$$

$$Q_{66} = G_{12} \tag{III.57}$$

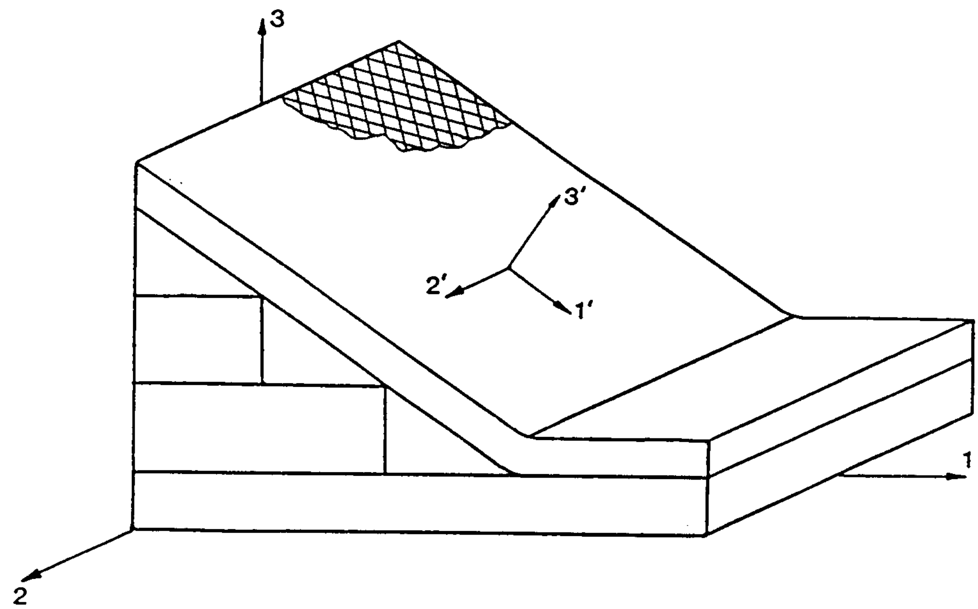

Figure III.4:

$$V = (1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{12}\nu_{23}\nu_{31})^{-1} \quad \text{(III.58)}$$

The presence of angle plies in the belt region making an angle θ in the $1'2'$-plane results in the following constitutive relationship

$$\{\sigma'\} = [\overline{Q}] \{\epsilon'\} \quad \text{(III.59)}$$

where the transformed reduced stiffnesses $\overline{Q}_{ij}$ are given in terms of reduced stiffnesses Q_{ij} as

$$\overline{Q}_{11} = c^4 Q_{11} + 2c^2 s^2 Q_{12} + s^4 Q_{22} + 4c^2 s^2 Q_{66} \quad \text{(III.60)}$$

$$\overline{Q}_{22} = s^4 Q_{11} + 2c^2 s^2 Q_{12} + c^4 Q_{22} + 4c^2 s^2 Q_{66} \quad \text{(III.61)}$$

$$\overline{Q}_{12} = c^2 s^2 Q_{11} + (c^4 + s^4) Q_{12} + c^2 s^2 Q_{22} - 4c^2 s^2 Q_{66} \quad \text{(III.62)}$$

$$\overline{Q}_{66} = 4c^2 s^2 Q_{11} - 8c^2 s^2 Q_{12} + 4c^2 s^2 Q_{22} + 4(c^2 - s^2)^2 Q_{66} \quad \text{(III.63)}$$

$$\overline{Q}_{33} = Q_{33} \tag{III.64}$$

$$\overline{Q}_{13} = c^2 Q_{13} + s^2 Q_{23} \tag{III.65}$$

$$\overline{Q}_{23} = s^2 Q_{13} + c^2 Q_{23} \tag{III.66}$$

$$c = cos\theta$$

$$s = sin\theta$$

Any ply in the belt portion of the taper makes an angle β with the loading axis if it is in the uncracked belt portion and an angle α if it is in the cracked belt portion. By performing a rotation about the 2-axis, the stiffness along the loading axis , takes the form

$$\{\sigma\} = [C]\{\epsilon\} \tag{III.67}$$

where σ_{ij} and ϵ_{ij} are in 123-axis system and C_{ij} represent the elements of transformed stiffness matrix in this coordinate system.

Since we have assumed

$$u(x, z) = U(x) \tag{III.68}$$

and

$$w = 0 \tag{III.69}$$

For plane stress condition in 1-3 plane (i.e. $\sigma_{i2} = 0$; i =1,2,3) stress strain relations reduce to

$$\sigma_{11} = \left(C_{11} - C_{12}^2/C_{22}\right) \epsilon_{11} \tag{III.70}$$

where

$$C_{11} = \overline{c}^4\overline{Q}_{11} + 2\overline{c}^2\overline{s}^2\overline{Q}_{13} + \overline{s}^4\overline{Q}_{33} + \overline{c}^2\overline{s}^2\overline{Q}_{55} \tag{III.71}$$

$$C_{12} = \overline{c}^2\overline{Q}_{12} + \overline{s}^2\overline{Q}_{23} \tag{III.72}$$

$$C_{22} = \overline{Q}_{22} \tag{III.73}$$

where $\overline{c}$ and $\overline{s}$ are cosine and sine of the angle which the cracked and uncracked belt portions makes with the loading axis.

The coefficient of ϵ_{11} in Equation (III.70) represents the transformed axial stiffness. This value is used in the derivation of A_B and A_{B1}.